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NOTE

The Fractional-Step Method for the Navier–Stokes Equations on Staggered Grids: The Accuracy of Three Variations

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Time integration of the Navier–Stokes equations is often carried out by means of the fractional-step procedure whereby the momentum equations and some form of Poisson equation are solved separately at each time step. Fractional step methods may be implemented in iterative or non-iterative forms, with the non-iterative schemes offering the possibility of considerable increases in efficiency. In this note we compare the accuracy and efficiency of an iterative scheme with those of two non-iterative schemes. It will be shown that the non-iterative pressure correction method achieves the same order of accuracy as the iterative method with a considerable increase in efficiency.

The Navier–Stokes equations in unsteady incompressible non-dimensional form are

$$u_t + (u \cdot \nabla)u = -\nabla P + \tfrac{1}{Re}\nabla^2 u, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

where u is the velocity, P the pressure, and Re the Reynolds number.

The continuous equations are discretised using Adams–Bashforth for the advective terms and Crank–Nicolson for the diffusive terms, giving the system

$$\frac{v^{n+1} - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -Gp^{n+1} + \frac{1}{2Re}L(v^{n+1} + v^n), \tag{3}$$

$$Dv^{n+1} = 0, \tag{4}$$

where (v, p) are the discrete velocity and pressure, respectively, and H is the discrete

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advection operator, G the discrete gradient, L the discrete Laplace operator, and D the discrete divergence. This is a discretisation that is second order in time, using an explicit scheme for the advection terms and an implicit scheme for the diffusion terms. Fractional-step methods integrate Eqs. (3) and (4) in a segregated manner; that is, the momentum equations are first solved for the velocity, and some form of Poisson equation is then solved for the pressure. The Poisson equation is constructed from the momentum equation and the continuity equation and, as well as providing the pressure, acts to enforce continuity.

Iterative method. In this method Eq. (3) is solved, using the best current value for p^{n+1}, to obtain v^*, an approximation to v^{n+1}. This approximate velocity will not initially satisfy continuity. A correction is then applied of the form

$$v^{n+1} = v^* - \Delta t G\pi, \tag{5}$$

where π is a pressure correction, such that the resulting v^{n+1} does satisfy continuity. An equation for π is constructed by substituting Eq. (5) into the continuity equation (4), to give

$$L\pi = Dv^*/\Delta t.$$

Once π is obtained, the velocity is corrected and the pressure is updated using the pressure correction as

$$p^{n+1} = p^{n+1} + \pi. \tag{6}$$

Equation (3) is then solved again using the updated pressure to obtain a new estimate of the velocity at the $n+1$ time level, and that velocity again corrected to enforce continuity and provide a pressure correction. This process is repeated until the divergence of the velocity after the solution of Eq. (3) satisfies a predefined value. The solution is then said to be converged and the integration continues to the next time step. For the first iteration at each time step p^{n+1} is set equal to p^n.

At the completion of the time step the solution will satisfy both Eq. (3) and Eq. (4), to within the required convergence condition, and is therefore expected to be second-order accurate in time.

Projection method. In this method Eq. (3) without the pressure gradient term is solved to obtain v^*. A correction is applied to this approximate velocity field of the form

$$v^{n+1} = v^* - \Delta t G\Phi \tag{7}$$

such that the resulting v^{n+1} does satisfy continuity. An equation for Φ is constructed by substituting Eq. (7) into Eq. (4), to give

$$L\Phi = Dv^*/\Delta t. \tag{8}$$

Once Φ is obtained, the velocity is corrected and the integration continues to the next time step. In the projection method it is necessary to solve the Poisson equation for Φ very accurately to ensure that the velocity remains divergence free. The projection method is a one-step method, and is therefore likely to require less computer time per time step than the iterative method. However, interaction between the implicitly discretised viscous terms and Φ leads to a first order in time error when standard boundary conditions are used [9].

Methods of this type were first suggested by Harlow and Welch [4] and Chorin [2]. The basic projection method was later modified for use with finite volumes defined on a staggered grid by Kim and Moin [6], and has since been used by many researchers for the simulation of unsteady flows (see Zang *et al.* [11] for a brief review and an application to non-staggered grids). Much effort has been expended on developing appropriate boundary conditions for the intermediate velocity field and the pressure Poisson equation to prevent the first-order error noted above (Kim and Moin [6], Karniadakis *et al.* [5]), while Perot [9] suggested a modified LU factorisation scheme which required no boundary conditions for the intermediate velocity and pressure.

Pressure correction method. The pressure correction method is identical to the iterative method, but with only a single iteration carried out at each time step. The discrete momentum equation, including the nth time level pressure, is solved to obtain v^*, as with the first iteration of the iterative method. The v^* field is then corrected to satisfy continuity and the pressure is corrected exactly as in the iterative method, but with only a single iteration at each time step. The interaction between the viscous terms and Φ now results in an error that is second order in time, and therefore does not adversely affect the overall accuracy of the scheme.

Methods of this type have been suggested by a number of authors (Van Kan [10], Bell and Colella [1]), while a similar approach has also been used with an approximate factorisation method (Dukowicz and Dvinsky [3]).

Discretisation and boundary conditions. The above schemes are defined on the standard MAC staggered grid using finite volumes, with standard second-order central differences used for the viscous terms, the pressure gradient, and divergence terms. The QUICK third-order upwind scheme is used for the advective terms (Leonard [7]). The momentum equations are inverted using an ADI scheme. Four sweeps of the ADI scheme were found to be sufficient to obtain accurate solutions for all the cases considered. A preconditioned restarted GMRES method is used to solve the Poisson Φ and pressure correction equations for all the methods. Other solvers, such as preconditioned conjugate gradient, incomplete LU, ADI, and Jacobi, have also been tested and found not to affect the overall accuracy or relative performance of the methods. Of the solvers tested GMRES was found to be the most efficient. Dirichlet velocity boundary conditions are set on all boundaries. The intermediate velocity v^* is set to the same boundary conditions as the velocity v. The use of a staggered grid with a finite volume method means that no boundary conditions are required for pressure. A zero normal gradient at all boundaries is specified for Φ and the pressure correction.

Results have been obtained for driven cavity flow (cf. Perng and Street [8]). Initially the fluid in the square cavity is quiescent. At time $t = 0$ the tangential velocity on the upper boundary is set to one, with the normal velocity on the upper boundary set to zero, and the other boundaries non-slip. The control parameter is the Reynolds number, which is set to $Re = 400$, based on the height of the cavity and the tangential velocity at the upper boundary. Solutions have been obtained on a 50×50 uniform grid. The solution is integrated in time from $t = 0$ to $t = 2$, in non-dimensional units, for time steps ranging from $\Delta t = 0.025$ to $\Delta t = 0.003125$. Times have been non-dimensionalised by the height of the cavity and the tangential velocity on the upper boundary. The error was quantified by obtaining the L_2 norm of the difference of the test solution and a solution that was obtained with $\Delta t = 0.0015625$ integrated for the same amount of time. The largest timestep, $\Delta t = 0.025$, is close to the empirically obtained stability limit of $\Delta t = 0.03$.

Figures 1 to 3 contain the errors for the pressure and U- and V-velocity components for the projection, pressure correction, and iterative schemes. First-order accuracy for the

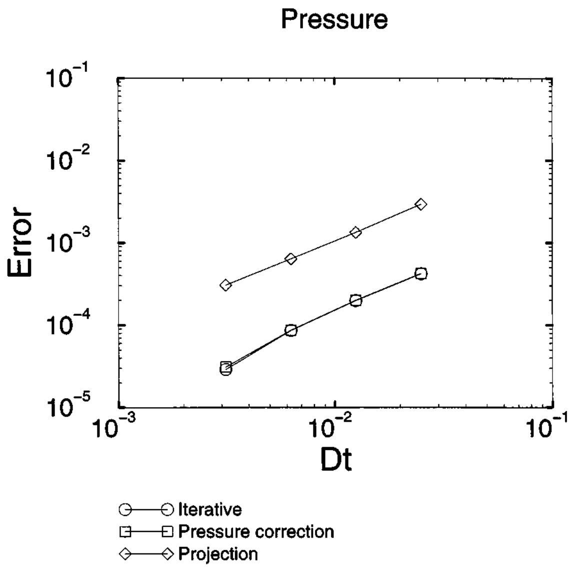

FIG. 1. Pressure error variation with time step.

pressure is obtained with all methods, with the pressure correction and iterative results almost identical and the projection method error an order of magnitude larger. Perot [9] observed that the pressure will always be first order in time for schemes of this type. Results for the U-velocity show that the projection method is first-order accurate while the pressure correction and iterative methods are second-order accurate. The iterative method gives results almost identical to those of the pressure correction method, with the projection method error more than an order of magnitude larger. Results for the V-velocity are similar

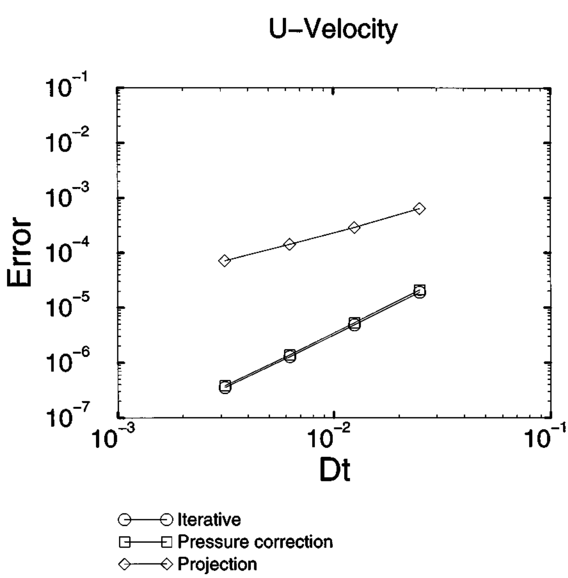

FIG. 2. U-velocity error variation with time step.

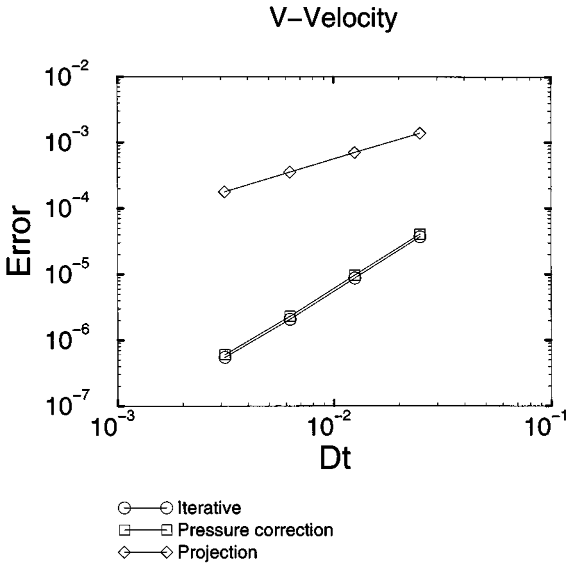

FIG. 3. V-velocity error variation with time step.

to those for the U-velocity. Alternative maximum and pointwise norms have also been tested and show the same behaviour as the L_2 norm.

Run times versus error graphs for each of the methods are presented in Fig. 4. The error is the average of the U-velocity and V-velocity errors, and the run time is in CPU seconds on a DEC Alpha 3000/700. The most efficient is the pressure correction scheme, requiring considerably less CPU time to achieve the same accuracy as the next most efficient iterative scheme. The projection method is the least efficient of the schemes by a considerable margin.

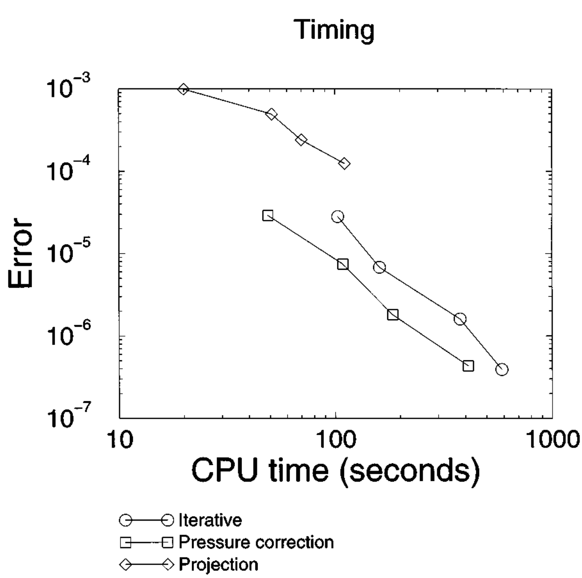

FIG. 4. Timing results.

Results have also been obtained with the Reynolds number $Re = 5000$. The order of convergence with respect to the time step and timing results for the high Reynolds number flow showed the same behaviour as those of the low Reynolds number flow.

The pressure correction method is the most efficient of those tested for this flow. The projection method is only first order in time and is significantly less efficient than both the pressure correction and iterative methods. The accuracy of the pressure correction method is very close to that of the iterative method, but requires considerably less CPU time per time step owing to its non-iterative form. A major advantage of the pressure correction method, when compared to other non-iterative projection-like methods, is that second-order accuracy has been achieved with the standard boundary conditions.

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